MATHEMATICS

SPECTRAL CHARACTERIZATION OF FINITE-DIMENSIONAL ALGEBRAS

BY

RUDOLF A. HIRSCHFELD AND BARRY E. JOHNSON [1]

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1. Introduction

Any algebra of endomorphisms of a finite-dimensional vector space has the "global finiteness property" that it is a finite-dimensional algebra, as well as the "local finiteness property" that each member has only a finite number of distinct points in its spectrum. Whereas the former property implies the latter, the converse implication is not generally true. In fact, consider the following two examples: (i) the convolution Banach algebra $L^1(0, 1)$, (ii) the algebra of all infinite matrices with only finitely many complex non-zero entries. Both are infinite-dimensional, but they do possess the "local finiteness property" (every element of $L^1(0, 1)$ is quasi-nilpotent relative to convolution). The first algebra is not semi-simple and the second one is not topologically complete. This sets our tune: are the semi-simple, topologically complete complex algebras eligible for the implication "local" $\Rightarrow$ "global" to hold?

Although by no means necessary, we shall restrict ourselves to the familiar class of Banach algebras.

Theorem. *Let A be a complex Banach algebra for which every element has a finite spectrum. Then, A is finite-dimensional modulo its radical.*

Recall that the Jacobson radical Rad (A) can be said to consist of those $q \in A$ for which qa is quasi-nilpotent (*i.e.* has spectrum $= \{0\}$) for every $a \in A$. Without loss of generality we may assume that A is semi-simple and that it has an identity 1.

In the events where A is commutative or C^*, very simple proofs can be given. For the general situation there is an enticing Baire category argument (which is supported by the fact that the local finiteness property implies that the spectrum depends continuously on the algebra element), but we have not even been able to establish along this line the existence of an upper bound for the order of the spectra. The key word in the following proof is: minimal idempotent. There is only a finite set $\{e_1, e_2, \ldots, e_n\}$ of these, and we have $\dim e_i A e_j < 1$. Despite the close

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resemblance to the Wedderburn-Artin structure theory of rings, the present minimal idempotents are *not* defined by the requirement that Ae_i be a minimal left ideal (since it is not known *a priori* that A has a socle), but rather by the customary partial ordering of projections.

A general reference is [1].

2. Proof of the theorem

(a) Let $E \subset A$ be a maximal set of commuting non-zero idempotents. The existence of E is guaranteed by the Lemma of Zorn and we have $1 \in E$. We are going to prove that E is *finite*. In fact, if not, there would be an infinite sequence $\{f_1, f_2, \ldots\}$ in E. Let

$$g_1 = f_1,\ g_2 = \sup(f_1, f_2) = f_1 + f_2 - f_1 f_2,\ g_{n+1} = \sup(g_n, f_{n+1}).$$

Each g_n is an idempotent, commuting with all of E and, hence, belongs to this maximal set E. The same holds for their differences

$$h_n = g_{n+1} - g_n,$$

for which we now have obtained the orthogonality relation

$$h_n h_m = 0 \qquad n \neq m,$$

whereas $h_n \neq 0$ $(n = 1, 2, \ldots)$.

Whether or not $\sup \|h_n\| < \infty$, we can always take a sequence $\{\lambda_n\}$ of non-zero complex numbers for which $\sum |\lambda_n| \cdot \|h_n\| < \infty$. Consequently,

$$a = \sum_{n=1}^{\infty} \lambda_n h_n$$

belongs to A. Since $ah_m = \lambda_m h_m$ or $(a - \lambda_m 1)h_m = 0$, each λ_m must belong to the spectrum of a, violating the finite-spectrum-assumption. Hence, E is a finite set.

(b) Let E be partially ordered by saying that $f \leqslant g$ if $fg = gf = f$. Since E is finite, we are able to indicate those elements $e_1, \ldots, e_n$ which are minimal relative to this ordering. We have both $e_i e_j \leqslant e_i$ and $e_i e_j \leqslant e_j$, whence $e_i e_j = 0$ if $i \neq j$. Moreover, every idempotent $e \in E$ is a sum of minimal ones, in particular,

$$e_1 + e_2 + \ldots + e_n = 1.$$

In fact, let $e_{i_1}, \ldots, e_{i_k}$ be those minimal elements which are less than or equal to $e \in E$. Then, $e' = e - (e_{i_1} + \ldots + e_{i_k})$ has the property that $e_i e' = 0$ for all $i = 1, \ldots, n$, whence $e' = 0$.

This decomposition of the identity leads to

$$A = Ae_1 + \dots + Ae_n,$$

but the Ae_i still not being manageable, we keep splitting

$$A = (e_1 + \dots + e_n)\, Ae_1 + \dots + (e_1 + \dots + e_n)\, Ae_n,$$

or

$$A = \sum_{i,j=1}^{n} e_i A e_j.$$

The rest of the proof consists in showing that the constituents e_iAe_j have finite dimensions.

(c) First, e_iAe_i is a Banach algebra with e_i as identity and each $a \in e_iAe_i$ has a finite spectrum by virtue of the well-known equality ([1], p. 35, Thm. (1.6.15))

$$\sigma_{e_i A e_i}(a) \cup \{0\} = \sigma_A(a)$$

(unless $n=1$, when $e_i = 1$). However, much more can be said: every a in e_iAe_i has a spectrum consisting of precisely one point. In fact, if $\sigma(a)$ would contain the two points λ and μ, upon integrating along a small circle around λ with μ in its exterior, we would obtain an idempotent

$$p = -\frac{1}{2\pi i} \oint (a - \zeta 1)^{-1} d\zeta$$

in e_iAe_i, different from 0 and e_i, which would contradict the minimality of e_i.

Consequently, every a in e_iAe_i has the form

$$a = \lambda e_i + q,$$

where λ is the single element of $\sigma(a)$ and q is a quasi-nilpotent in e_iAe_i.

We are going to show that e_iAe_i possesses no non-zero quasi-nilpotents, though, whence $e_iAe_i = \mathbf{C}\, e_i$ and $\dim(e_iAe_i) = 1$. We do this by proving that any quasi-nilpotent q in e_iAe_i must belong to Rad (A). The latter being zero by semi-simplicity, this does imply that $q = 0$. Thus, we have to verify the equality

$$\lim_{n\to\infty} \|(qa)^n\|^{1/n} = 0 \quad \text{for any } a \in A.$$

Using that $q = e_iq = qe_i = e_iqe_i$, we can write

$$\begin{aligned}(qa)^n &= (qe_i a)(e_i qe_i a)(e_i qe_i a) \dots (e_i qe_i a)(e_i qa)\\ &= (qe_i ae_i)(qe_i ae_i) \dots (qe_i ae_i)(qa)\\ &= (qe_i ae_i)^{n-1}(qa)\\ &= (qb)^{n-1}(qa).\end{aligned}$$

The element $b = e_iae_i$ featuring here belongs to e_iAe_i, so that we have reduced the problem from A to e_iAe_i. That is, it is sufficient to show that

$$\lim \|(qb)^n\|^{1/n} = 0 \quad \text{for every } b \in e_iAe_i.$$

In other words, it remains to prove that *every* quasi-nilpotent $q \in e_iAe_i$ belongs to Rad (e_iAe_i). Notice first that every singular element of e_iAe_i, having spectrum $=\{0\}$, is quasi-nilpotent. Starting with a regular $b \in e_iAe_i$, we see that bq, being a product of a regular and a singular element, is singular, whence quasi-nilpotent. Next, if b happens to be singular, or quasi-nilpotent, in e_iAe_i, we obtain a regular element $b_\lambda = \lambda e_i + b$ for every $\lambda \neq 0$, for which qb_λ is singular by the previous argument. The set of all singular elements being closed in a Banach algebra, we conclude that $qb = \lim\limits_{\lambda \to 0} qb_\lambda$ is singular, or quasi-nilpotent. This completes the proof that $\dim e_iAe_i = 1$.

(d) Finally, we consider the Banach space e_iAe_j, where $i \neq j$. If this space equals $\{0\}$, it has dimension 0 and we are done. If e_iAe_j happens to be $\neq \{0\}$, we will exhibit a linear isomorphism

$$R_j^i\colon e_iAe_i \to e_iAe_j.$$

By virtue of (c) this entails $\dim (e_iAe_j) = 1$.

In the first place, let $u \neq 0$ belong to e_iA. We claim that $uA = e_iA$. For one thing, we may write $u = e_iv$, whence $uA = e_ivA \subset e_iA$. For another, we show that $e_iA \subset uA$ in the following fashion. Suppose, indirectly, that $ute_i = 0$ for all $t \in A$. Then, $(ut)^2 = ut \cdot ut = (ute_i)vt = 0$, so that ut is nilpotent for all $t \in A$, whence $u \in \mathrm{Rad}\,(A) = \{0\}$, contradicting the assumption that $u \neq 0$. Consequently, there exists $w \in A$ such that $uwe_i \neq 0$. Now, this element $uwe_i = e_ivwe_i$ belongs to $e_iAe_i = \mathbf{C}\, e_i$ by (c) and, hence, can be written as $uwe_i = \lambda e_i$, where $\lambda \neq 0$ in $\mathbf{C}$. Therefore, $e_i = u(\lambda^{-1}we_i)$ and so $e_iA = u(\lambda^{-1}we_i)A \subset uA$. The statement $uA = e_iA$ now follows.

Assume now that e_iAe_j contains a non-zero element a. Then a is *a fortiori* a non-zero member of e_iA, whence $e_iA = aA$ by what we proved above and so $e_i \in aA$. Hence, there exists $a_0 \in A$ such that $e_i = aa_0$. We may write $a = e_ite_j$, $t \in A$, whence $e_i = (e_ite_j) \cdot (e_ja_0e_i) = aa'$, where $a' = e_ja_0e_i$ is a non-zero element of e_jAe_i. Repeat this argument with a' in place of a, to the effect that there exists $b \in e_iAe_j$ with $a'b = e_j$. We have

$$a = ae_j = aa'b = e_ib = b,$$

which says that a intertwines e_i and e_j in the sense that

$$e_ia = ae_j.$$

We now let a act as the right multiplicator

$$R_j^i\colon x \mapsto xa$$

on e_iAe_i. Its range belongs to e_iAe_j, because $R_j^ix = R_j^i(xe_i) = xe_ia = xae_j = e_ixae_j$.

Similarly, $y \mapsto ya'$ defines a linear map

$$R_i^j\colon e_iAe_j \to e_iAe_i.$$

By their very construction, these maps satisfy

$$R_j^i R_i^j = I \text{ on } e_i A e_j, \quad R_i^j R_j^i = I \text{ on } e_i A e_i,$$

showing that R_j^i is an isomorphism, indeed.

We conclude the proof by the remark that $A = \sum_{i,j} e_i A e_j$ implies dim $A < n^2$.

University of Hawaii
Yale University and
University of Newcastle

REFERENCE

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